Engineering Notes

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Stability of a Stack of Spinning Bodies

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Introduction

MANY papers (see Refs. 1-4, to quote only a few) have been written on the nutational stability of single or dual spinners containing dampers, based on the Routh stability criterion and/or energy considerations. Papers following the former approach usually start off very rigorously, by considering the effect of motion of the damper mass (masses) on the inertia tensor(s) and on the position of the center of mass of the system. They may also begin by considering general cases where the inertia tensors are triaxial, the bearing axis does not coincide with one of the inertia ellipsoid principal axes and, possibly, sloshing liquids are present. Very soon the equations become intractable and, therefore, at some point, drastic simplifications are required. In any case, application of the Routh stability criterion requires linearization. The influence of the simplifications is not always clear and the complexity of the equations excludes the majority of engineers from getting a physical feel of the phenomena. Papers based on the so-called "energy sink" concept are clear when applied to the single spinning body, but become somewhat foggy when dual spinners containing a servo loop are treated; in any case, they do not relate power dissipation to damper characteristics and spacecraft motion.

The approach that will be used herein aims at giving a physical insight into the damping phenomenon and to give stability conditions (which turn out to be very straightforward) for the more general case of any number of bodies spinning around a single axis, carrying any number of dampers. It will also provide damper design optimum parameters. In order to reach the aforementioned scope, a situation making the mathematics comparatively easy is studied, the single real cases being approximations to the ideal case. The approach consists in finding: 1) the motion of the bodies forming the stack when no dampers are present; 2) the damper masses motion forced by the body motion determined by point 1; and, finally, 3) their effects on the spin axis motion.

Description of the System to be Studied

The physical composition of the system (see Fig. 1) is assumed as being a stack of N rigid bodies, with axisymmetric inertia ellipsoids, which may spin around a common axis, coincident with the symmetry axis of the inertia ellipsoids. Each body may rotate around the stack axis at a different speed. Attached to the bodies it is assumed there are M dampers, each of which is considered to be composed of a point mass free to move parallel to the stack symmetry axis at a distance $R \neq 0$, a spring, and a dashpot.

We shall further restrict our investigation to the case where the stack axis does not change its orientation very much during the motion of the stack and where no external torques act on it. The mass and displacement (relative to the body) of each damper are so small that it may be assumed that the position of the center of mass and the inertia ellipsoid of the stack are determined by the bodies alone. In line with this assumption, it may be figured that the motion of the stack is affected by the relative motion of the dampers masses moving with respect to the bodies in the long run only. In other words, the damping "time constant" is large as compared to the "periods" of other phenomena (e.g., the damper mass relative motion).

Stack Motion

With no external torque acting on the system, the following relationship holds in general

$$\bar{K}_0 = \bar{K} + \sum_{I}^{M}{}_{j} \; m_j \overline{OP}_j \times \dot{\overline{OP}}_j = \text{const} \qquad (1)$$

where $\bar{K}_0$ is the moment of momentum vector of the system, referred to its center of mass O, $\bar{K}$ the moment of momentum vector of the stack referred to O, m_j the mass of the jth damper and P_j its position.

As stated in the introduction, we shall first find the motion of the bodies forming the stack when no dampers are present. If the masses of the dampers are small, as it is usually the case, compared with the masses of the bodies, one may ignore the second term on the right-hand side of Eq. (1) and, to a first approximation, one may write

$$\bar{K}_0 = \bar{K} \qquad (2)$$

With respect to a reference frame defined by the unit vectors $\bar{u}_1$, $\bar{u}_2$, and $\bar{u}_3$, chosen so that $\bar{u}_3$ coincides with the geometrical axis of the stack, and the $\bar{u}_1$ axis is perpendicular to $\bar{K}_0$ (the direction of $\bar{u}_1$ is defined when the directions of u_3 and $\bar{K}_0$ do not coincide), one may write[5]

$$\bar{K} = A\dot{\theta}\bar{u}_1 + A\dot{\psi}\sin\theta\bar{u}_2 + \sum_{I}^{N}{}_{i} \; C_i(\dot{\varphi}_i + \dot{\psi}\cos\theta)\bar{u}_3 \qquad (3)$$

where A is the central moment of inertia of the whole stack with respect to any axis perpendicular to the symmetry stack axis; C_i is the central moment of inertia of the ith body with respect to the stack axis; θ is the angle between $\bar{u}_3$ and $\bar{K}_0$; ψ is the angle swept by the plane defined by $\bar{u}_1$ and $\bar{K}_0$ around $\bar{K}_0$, starting from an inertially fixed reference plane containing $\bar{K}_0$; and φ_i is the angle swept by a plane attached to the ith body and containing $\bar{u}_3$, starting from the plane defined by $\bar{u}_3$ and $\bar{u}_1$.

Due to Eqs. (2) and (3) and the definition of the $\{\bar{u}_1, \bar{u}_2, \bar{u}_3\}$ frame, $\bar{K}\cdot\bar{u}_1$ is zero, while $\bar{K}\cdot\bar{u}_2$ is equal to $K_0\sin\theta$. Therefore

$$A\dot{\theta} = 0 \Rightarrow \theta = \text{const} \qquad (4)$$

$$A\dot{\psi}\sin\theta = K_0\sin\theta \Rightarrow \dot{\psi} = K_0/A \qquad (5)$$

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Thus, the *motion of the stack axis is independent of the presence of internal torques between the bodies. The stack axis describes a cone centered around the moment of momentum vector of the stack; the angular rate of the stack axis around the moment of momentum vector is a constant that equals the moment of momentum of the whole stack over the transversal moment of inertia of the stack.*

With applications in mind, it is reasonable to assume that net torques due to friction and servomotors are zero on the average and that their fluctuations are filtered by the large moments of inertia of the bodies. The third Euler equation applied to the ith body then implies

$$C_i(\dot\varphi_i + \dot\psi\cos\theta) = \text{const} \qquad (i=1,\ldots,N) \tag{6}$$

By combining Eqs. (4-6) one has the result

$$\dot\varphi_i = \text{const} \qquad (i=1,\ldots,N) \tag{7}$$

Damper Masses Motion

We shall now find the motion of the damper masses, forced by the previously determined motion of the stack. Since they are unnecessary, subscripts i and j will be dropped in this paragraph. Dynamic equilibrium of any one of the damper masses requires that the following relationship be satisfied:

$$m\ddot{\overline{OP}}\cdot\bar u_3 = -ks - h\dot s \tag{8}$$

where $\overline{OP}$ is the position vector of the damper mass m, k and h are the elastic and viscous constants of the damper, and s is the displacement of the damper mass from its mean position.

Let L be the distance of this mean position from the plane containing O and perpendicular to the stack axis, and R the distance from said axis. If the plane containing the damper under consideration is used to measure angle φ, defining the position of the body carrying the damper around the stack axis, one has

$$\overline{OP} = R\cos\varphi\bar u_1 + R\sin\varphi\bar u_2 + (L+s)\bar u_3 \tag{9}$$

and, taking into account that the angular velocity of the $\{\bar u_1,\bar u_2,\bar u_3\}$ frame with respect to inertial space is given by $\dot\psi(\sin\theta\bar u_2 + \cos\theta\bar u_3)$, by differentiating Eq. (9), one obtains

$$\dot{\overline{OP}} = [(L+s)\dot\psi\sin\theta - R(\dot\varphi + \dot\psi\cos\theta)\sin\varphi]\bar u_1 + R(\dot\varphi+\dot\psi\cos\theta)\cos\varphi\bar u_2 + (\dot s - R\dot\psi\cos\varphi\sin\theta)\bar u_3 \tag{10}$$

By differentiating again and projecting onto $\bar u_3$, one obtains

$$\ddot{\overline{OP}}\cdot\bar u_3 = \ddot s - (L+s)\dot\psi^2\sin^2\theta + R(2\dot\varphi + \dot\psi\cos\theta)\dot\psi\sin\varphi\sin\theta \tag{11}$$

Since we are interested in finding the effect of the damper when θ is small, one may simplify Eq. (11), thus getting

$$\ddot{\overline{OP}}\cdot\bar u_3 = \ddot s + \theta R(2\dot\varphi + \dot\psi)\dot\psi\sin\varphi \tag{12}$$

Substitution of Eq. (12) into Eq. (8) yields

$$\ddot s + (h/m)\dot s + (k/m)s = -\theta R(2\dot\varphi+\dot\psi)\dot\psi\sin\varphi \tag{13}$$

In line with the assumption that the effect of the damper is a slight perturbation of the stack motion, only the steady solution of Eq. (13) describing the damper motion need be considered

$$s = \theta S\sin(\varphi+\alpha) \tag{14}$$

where

$$S = -\frac{mR\dot\psi(2\dot\varphi+\dot\psi)}{[(k-m\dot\varphi^2)^2 + h^2\dot\varphi^2]^{1/2}} \qquad \alpha = \tan^{-1}\frac{h\dot\varphi}{m\dot\varphi^2 - k} \tag{15}$$

Equation (14) and its derivative

$$\dot s = \theta\dot\varphi S\cos(\varphi+\alpha) \tag{16}$$

will be employed to determine the effect of the damper masses on the motion of the stack.

Effect of Dampers Motion on Stack Motion

From Eqs. (9) and (10), it follows

$$\begin{aligned}\overline{OP}\times\dot{\overline{OP}} &= R[(\dot s - R\dot\psi\cos\varphi\sin\theta)\sin\varphi - (L+s)(\dot\varphi+\dot\psi\cos\theta)\cos\varphi]\bar u_1 \\ &+ \{R(R\dot\psi\cos\varphi\sin\theta - \dot s)\cos\varphi + (L+s)[(L+s)\dot\psi\sin\theta \\ &- R(\dot\varphi+\dot\psi\cos\theta)\sin\varphi]\}\bar u_2 + R\{R(\dot\varphi+\dot\psi\cos\theta)\cos^2\varphi \\ &- [(L+s)\dot\psi\sin\theta - R(\dot\varphi+\dot\psi\cos\theta)\sin\varphi]\sin\varphi\}\bar u_3\end{aligned} \tag{17}$$

If second-order terms in θ are ignored and Eqs. (14) and (16) are used, Eq. (17) becomes

$$\begin{aligned}\overline{OP}\times\dot{\overline{OP}} = R\{&\{[S\dot\varphi\cos(\varphi+\alpha) - R\dot\psi\cos\varphi]\sin\varphi \\ &- S(\dot\varphi+\dot\psi)\sin(\varphi+\alpha)\cos\varphi\}\theta - L(\dot\varphi+\dot\psi)\cos\varphi\}\bar u_1 \\ &+ \{[R^2\dot\psi\cos^2\varphi - RS\dot\varphi\cos(\varphi+\alpha)\cos\varphi + L^2\dot\psi \\ &- RS(\dot\varphi+\dot\psi)\sin(\varphi+\alpha)\sin\varphi]\theta - L(\dot\varphi+\dot\psi)\sin\varphi\}\bar u_2 \\ &+ [(RL\dot\varphi\sin\varphi)\theta + R^2(\dot\varphi+\dot\psi)]\bar u_3\end{aligned} \tag{18}$$

The preceding expression is valid for each damper. Therefore, by substituting Eqs. (3) and (18) into Eq. (1), when considering the component on $\bar u_1$ and remembering that $\bar K_0\cdot\bar u_1$ is zero, one may write

$$\begin{aligned}A\dot\theta + \sum_{1}^{M}{}_j\, m_jR_j\{&\{[S_j\dot\varphi_j\cos(\varphi_j+\alpha_j) - R_j\dot\psi\cos\varphi_j]\sin\varphi_j \\ &- S_j(\dot\varphi_j+\dot\psi_j)\sin(\varphi_j+\alpha_j)\cos\varphi_j\}\theta - L_j(\dot\varphi_j+\dot\psi)\cos\varphi_j\} = 0\end{aligned} \tag{19}$$

In Eq. (19) $\varphi_j,\dot\varphi_j$ refers to the body carrying the jth damper. Therefore, if the same body carries $Q>1$ dampers, the same values of $\varphi,\dot\varphi$ will appear in Q terms of the summation.

In the introduction, it was stated that our study would be restricted to the case where the dampers affected the stack motion in the long run only. Therefore one can average Eq. (19) by treating $\theta,\dot\theta,\dot\varphi,\dot\psi$ as constants, which was shown to be the solution of the first approximation. By applying this procedure one gets

$$A\dot\theta - \left[\sum_{1}^{M}{}_j\,\frac{1}{2}m_jR_jS_j(2\dot\varphi_j+\dot\psi)\sin\alpha_j\right]\theta = 0 \tag{20}$$

and, by introducing Eqs. (15) and solving for $\dot\theta$

$$\dot\theta = \left\{\sum_{1}^{M}{}_j\,\frac{m_j^2R_j^2h_j\dot\psi\dot\varphi_j(2\dot\varphi_j+\dot\psi)^2}{2A[(k_j - m_j\dot\varphi_j^2)^2 + h_j^2\dot\varphi_j^2]}\right\}\theta \Rightarrow \theta = \theta_0 e^{\sigma t} \tag{21}$$

where θ_0 is the initial value of θ, and σ is given by

$$\sigma = \sum_{1}^{M}{}_j\,\frac{m_j^2R_j^2h_j\dot\psi\dot\varphi_j(2\dot\varphi_j+\dot\psi)^2}{2A[(k_j-m_j\dot\varphi_j^2)^2 + h_j^2\dot\varphi_j^2]} \tag{22}$$

It follows that the stability condition is

$$\boxed{\sigma < 0} \tag{23}$$

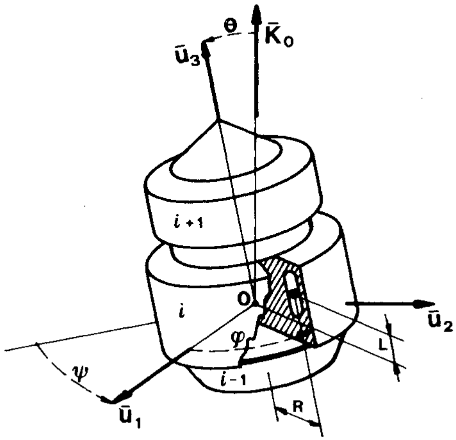

Fig. 1 System configuration, $\{\bar{u}_1,\bar{u}_2,\bar{u}_3\}$ frame and θ,ψ,φ angles.

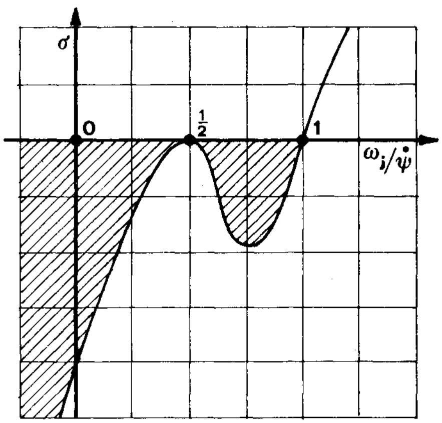

Fig. 2 Typical damper influence on nutational stability.

It may be reassuring to know that we developed the relationships resulting by considering the other two components of Eq. (1) in the same manner and that we found that they do not contain θ nor its derivatives and they lead to correct the values of $\dot{\psi}$ and $\dot{\varphi}$ found in the previous paragraph by a small constant amount, which reduces to zero together with the damper masses.

From a practical point of view it is convenient to eliminate angular rates $\dot{\varphi}_j$ (which may be defined only when $\theta \neq 0$) and introduce the nominal inertial angular rates ω_j that are related to $\dot{\varphi}_j$ by the expression $\omega_j = \dot{\varphi}_j + \dot{\psi}$ where, as shown,

$$A\dot{\psi} \cong K_0 \cong \sum_{1}^{N}{}_i\, C_i\omega_i$$

$$\sigma = \sum_{1}^{M}{}_j \frac{m_j^2 R_j^2 h_j \left(\frac{\omega_j}{\dot{\psi}} - 1\right)\left(2\frac{\omega_j}{\dot{\psi}} - 1\right)^2}{2A\left\{\left[\frac{k_j}{\dot{\psi}^2} - m_j\left(\frac{\omega_j}{\dot{\psi}} - 1\right)^2\right]^2 + \frac{h_j^2}{\dot{\psi}^2}\left(\frac{\omega_j}{\dot{\psi}} - 1\right)^2\right\}} \tag{24}$$

The typical behavior of each term of the summation in Eq. (24) is plotted in Fig. 2 as a function of the ratio $\omega_j/\dot{\psi}$.

As a conclusion, the following stability criterion may be formulated: *A damper—composed of a mass free to move parallel to the stack axis, a spring, and a dashpot, and mounted on one of the bodies forming a stack of spinning symmetrical rigid bodies—tends to increase the nutation angle if the body rotates at a rate higher than the nutational rate (i.e., the axial moment of momentum of the stack over the transversal moment of inertia of the stack); it tends to reduce the nutation angle if the body rotates at a lower rate; it has no effect if the body rotates at either the nutational rate or half the nutational rate.*

References

[1]Landon, V.D. and Stewart, B., "Nutational Stability of an Axisymmetric Body Containing a Rotor," *Journal of Spacecraft and Rockets,* Vol. 1, Nov.-Dec. 1964, pp. 682-684.

[2]Likins, P.W., "Attitude Stability Criteria for Dual Spin Spacecraft," *Journal of Spacecraft and Rockets,* Vol. 4, Dec. 1967, pp. 1638-1643.

[3]Kaplan, M.H., *Modern Spacecraft Dynamics and Control,* Wiley, New York, 1976, Chap. 5.2.

[4]Sen, A.K., "The Dynamic Stability of a Dual-Spin Satellite," *IEEE Transactions on Aerospace and Electronic Systems,* Vol. AE-13, July 1977, pp. 370-377.

[5]Levi-Civita, T. and Amaldi, U., *Lezioni di Meccanica Razionale,* Vol. 2, Pt. 2, Zanichelli, 1927.